Experiment: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$.

constants

ie. $x = au + bv$

$y = cu + dv$

$\int_2^7 K\,d\theta$.

$= 5K$

What is $dx \wedge dy$?

$\rightarrow \quad dx = d(au + bv)$

$dx = a\,du + b\,dv$

$dy = c\,du + d\,dv$

$dx \wedge dy = (a\,du + b\,dv) \wedge (c\,du + d\,dv)$

$= ad\; du \wedge dv + bc\; dv \wedge du$

$= ad\; du \wedge dv - bc\; du \wedge dv$

$= \boxed{(ad - bc)\; du \wedge dv.}$

Moral: variable $x_1, x_2, \ldots$

Substitute $x_1 = x_1(u_1, u_2, \ldots)$

$x_2 = x_2(u_1, u_2, \ldots)$

1 variable calculus. $\left(\begin{array}{l} x = x(u) \\ dx = x'(u)\,du \end{array} \right.$

Several variables

$$dx = x'(u)\,du$$

$$\begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} (x_1)_{u_1} & (x_1)_{u_2} & \cdots \\ (x_2)_{u_1} & (x_2)_{u_2} & \cdots \\ \vdots & & \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ \vdots \end{pmatrix}$$

matrix mult.

Volume form

$$dx_1 \wedge dx_2 \wedge \cdots = \det\left(x'(u)\right) \cdot du_1 \wedge du_2 \wedge \cdots$$

inside integral

$$dx_1\,dx_2 \cdots = \left|\det x'(u)\right| du_1\,du_2 \cdots$$

Example: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{pmatrix}$ ← new variables are ρ, θ, ϕ

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta\,d\rho - \rho\sin\phi\sin\theta\,d\theta + \rho\cos\phi\cos\theta\,d\phi \\ \\ \\ \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cdot\ - & - & \end{pmatrix}}_{M} \begin{pmatrix} d\rho \\ d\theta \\ d\phi \end{pmatrix}$$

$$dx \wedge dy \wedge dz = \underset{\uparrow \atop -\rho^2 \sin\phi}{(\det M)}\, d\rho \wedge d\theta \wedge d\phi$$

$$dx\, dy\, dz = \underbrace{|\det M|}_{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

Polar coordinate example

$$x = r\cos\theta = x(r,\theta)$$
$$y = r\sin\theta = y(r,\theta)$$

$$dx \wedge dy = \left| \begin{matrix} \text{derivative} \\ \text{matrix} \end{matrix} \right| dr \wedge d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr \wedge d\theta$$

$$= \left(\cos\theta\,(r\cos\theta) - (-r\sin\theta)\sin\theta\right) dr \wedge d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr \wedge d\theta$$

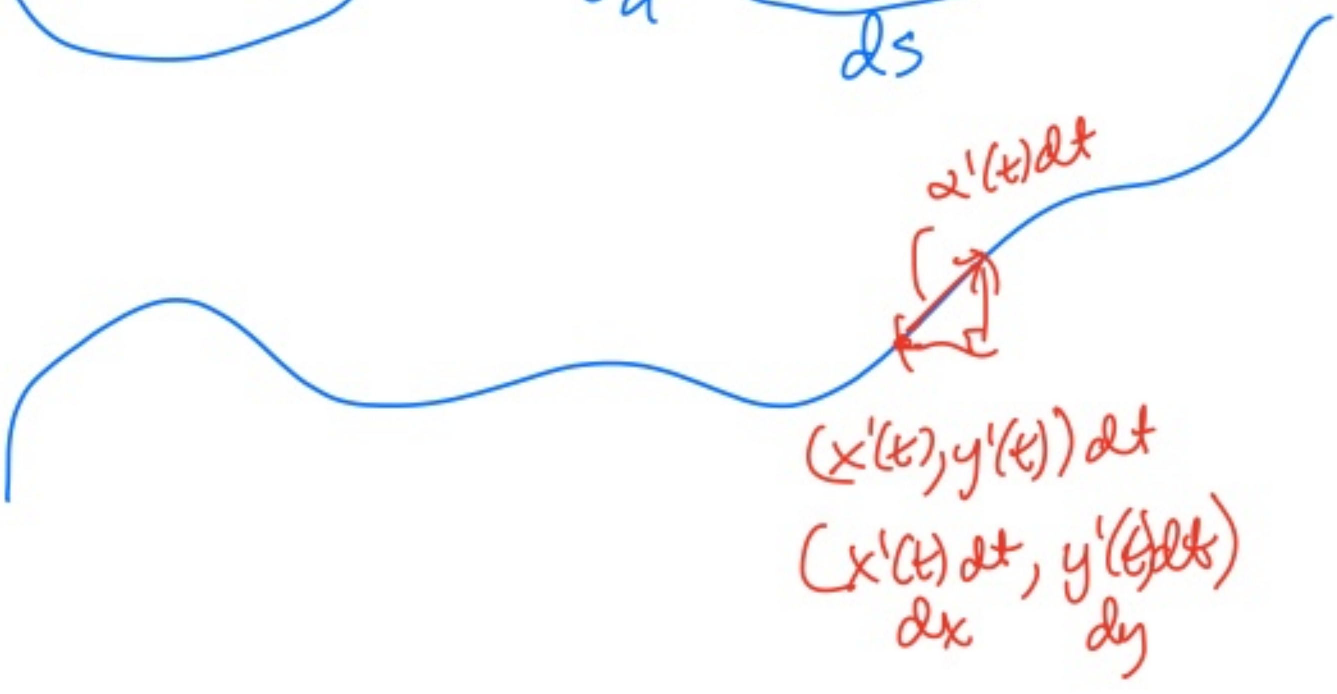
$$= r\, dr \wedge d\theta.$$

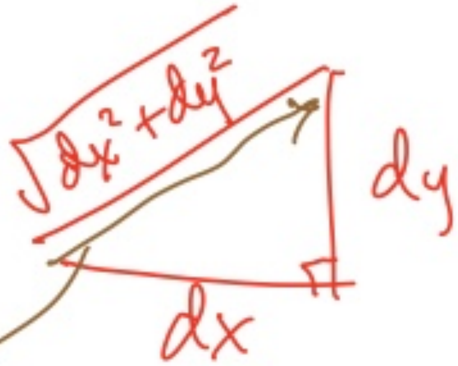
Integrals over curves & surfaces.

<u>Example</u>: Integrals over curves.

$$\alpha(t) = (x(t), y(t), \dots)$$

$$\boxed{\begin{array}{c}\text{Length} \\ \text{of } \alpha(t) \\ a \le t \le b\end{array}} = \int_a^b \underbrace{|\alpha'(t)|\, dt}_{ds}$$





$x = x(t)$

$y = y(t)$

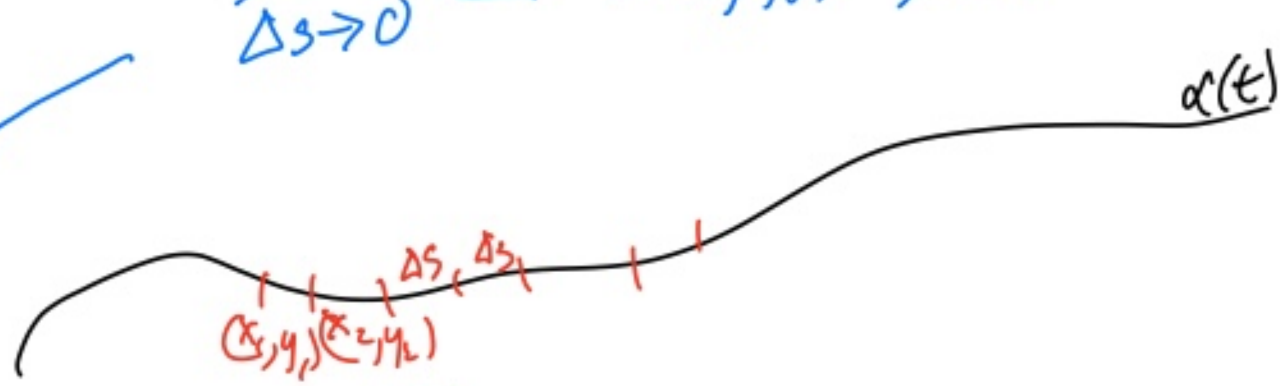
$dx = x'(t)\,dt$

$dy = y'(t)\,dt.$

$$\text{vector} = (dx, dy) = (x'(t)\,dt,\ y'(t)\,dt)$$

Integral of a scalar function $f(x, y, \cdots)$ over a curve $\alpha(t) = (x(t), y(t), \cdots)$ from $a \le t \le b$

$$\lim_{\Delta s \to 0} \sum f(x_j, y_j, \cdots)\, \Delta s$$



$$= \int_a^b f\big(\underset{x(t),\,y(t)}{\alpha(t)}\big) \cdot ds(t)$$

$$= \int_a^b f(\alpha(t)) \underbrace{\underbrace{|\underbrace{\alpha'(t)}_{\text{velo}}|}_{\text{speed}} dt}_{\substack{\text{distance.} \\ ds}}$$

Example of use:

density of curved rod is $\leftarrow$ gm/cm

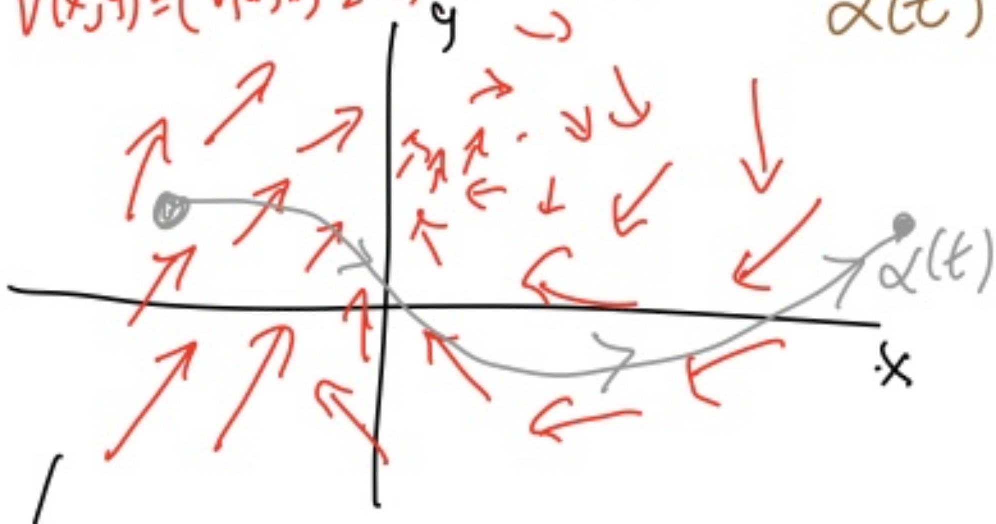
$f(x,y)$, rod $\alpha(t)$ $a \le t \le b$.

$$(\text{Mass of rod}) = \int_a^b f(\alpha(t)) \cdot \underline{|\alpha'(t)|\,dt}.$$

More useful kind of integral along a curve — "line integral"

Settings - Have a vector field $V(x,y,\ldots)$ and a curve $\alpha(t)$.

$V(x,y) = (V_1(x,y), V_2(x,y))$



$$\hookrightarrow \oint_\alpha V\cdot ds =$$ "line integral of the vector field $V(x,y)$ along the curve $\alpha = (x(t), y(t), \dots)$ from $a \le t \le b$"

$$= \int_{t=a}^{b} V(\alpha(t)) \bullet (dx, dy, \dots)$$

dot product

$$= \int_a^b V(\alpha(t)) \cdot (x'(t), y'(t), \dots)\, dt$$

Measures the total amount that the vector field is pushing along the curve in the direction of motion.

<u>Example of use:</u> $F(x,y,\dots) =$ Force field

$\alpha(t) \leftarrow$ position at time t.

$$\oint_\alpha F\cdot ds =$$ Work (energy) done by the force along the curve α.

example : if F = force of gravity on object.

$\alpha \leftarrow$ path of space ship.

$\Rightarrow \oint_{\alpha} F \cdot ds$ = amount of energy that gravity does along the curve α.

$\Rightarrow -\oint_{\alpha} F \cdot ds$ = amount of energy expended by the space ship.

Examples:

① $V(x,y) = (y+2x, y)$

$$\alpha(t) = (\overset{x(t)}{\cos(t)}, \overset{y(t)}{\sin(t)}) \quad 0 \le t \le 2\pi$$

Let's compute $\oint_{\alpha} V \cdot ds$. $\quad d\vec{s} = (x'(t), y'(t))$

$$\oint V \cdot ds = \int_0^{2\pi} \Big(\sin(t) + 2(\cos(t)), \sin(t)\Big) \cdot (-\sin(t), \cos(t))\, dt$$

$$= \int_0^{2\pi} \left(-\sin^2(t) - 2\cos(t)\sin(t)\right) + \sin(t)\cos(t) \quad dt$$

$$= \int_0^{2\pi} \left[-\sin^2(t) - \cos(t)\sin(t) \right] dt$$

$$\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$$

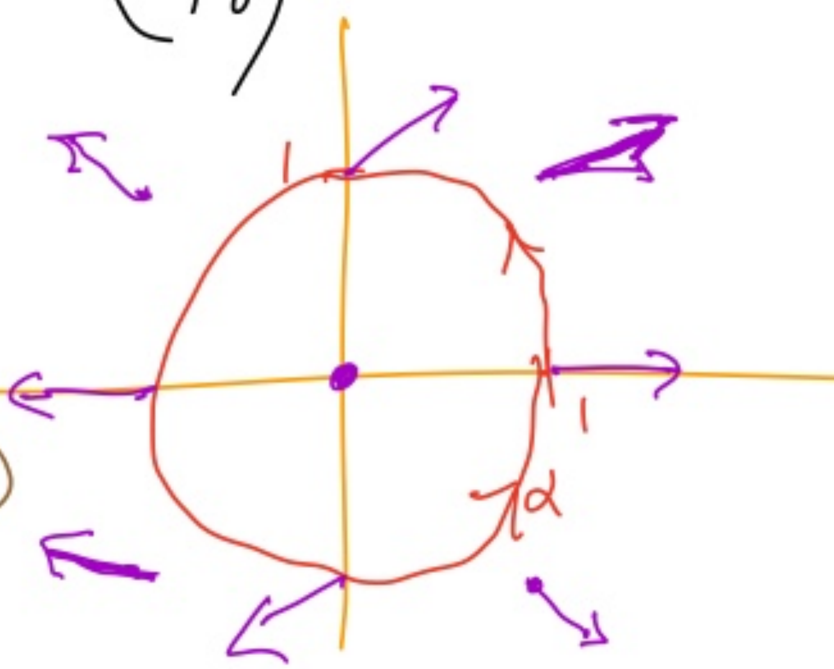
$$= \int_0^{2\pi} \left[-\frac{1}{2} + \frac{1}{2}\cos(2t) \right] dt + \int_0^{2\pi} -\cos(t)\sin(t)\, dt$$

$$u = \cos(t)$$
$$du = -\sin(t)\, dt$$

$$= \left(-\frac{t}{2} + \frac{1}{4}\sin(2t) \right)\Big|_0^{2\pi} + \underbrace{\int_1^1 u\, du}_{0}$$

$$= -\frac{2\pi}{2} + \underbrace{\frac{\sin(4\pi)}{4}}_{0} - (0 + 0)$$

$$= \boxed{-\pi}$$



(x,y)	$v(x,y) = (y+2x, y)$
$(0,0)$	$(0,0)$
$(1,1)$	$(3,1)$
$(-1,1)$	$(-1,1)$
$(+1,-1)$	$(1,-1)$
$(-1,-1)$	$(-3,-1)$

makes sense answer is negative!

(1,0)	(2,0)
(0,1)	(1,1)
(-1,0)	(-2,0)
(0,-1)	(-1,-1)